

Agenda

1. Building a 2-level model
2. Interpreting multi-level estimates
3. Specifying and and interpreting in R

Building a two-level model (redux)

Building a two-level model

Linear model of test scores

Scores depend on students' age, sex, race, and ethnicity.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Random intercept

Let each class have its own overall performance.

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Random slopes

Let the “effect” of students’ characteristics differ from classroom to classroom

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Intercept predictors

Does each classes' *average score* depend on classroom features? (number of students and teacher's experience)

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Building a two-level model

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \eta_{3k}$$

Slope predictors

Does the expected score difference between older and younger students depend on the size of the class?

Building a two-level model

Group-level covariates on slope and intercept

In most cases, covariates on slope models should be included in intercept model as well

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \\ & \gamma_{03} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Covariance model

Covariance structure

Covariance of the level-one coefficients modeled using variance Φ and correlations P .

$$\begin{bmatrix} \eta_{0k} \\ \eta_{1k} \\ \eta_{2k} \\ \eta_{3k} \end{bmatrix} \sim \text{MVNorm} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Phi P \Phi \right)$$

$$\Phi = \begin{bmatrix} \phi_0 & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & \phi_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{01} & 1 & \rho_{12} & \rho_{13} \\ \rho_{02} & \rho_{12} & 1 & \rho_{23} \\ \rho_{03} & \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

Building a two-level model

Complete model

The full two-level model describes variation in test scores as a function of student- and class-level covariates.

Residual variation is attributed to within-class (σ) and between-class (ϕ) differences.

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \\ & \gamma_{03} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Interpreting estimates

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \gamma_{03} \text{TeacherFemale}_k + \gamma_{04} \text{TeacherBlack}_k + \gamma_{05} \text{PropBlack}_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\beta_{3k} = \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \gamma_{32} \text{PropBlack}_k + \eta_{3k}$$

Expected score when all covariates are zero

Baseline effect of class size

Baseline effect of teacher experience

Baseline effect of age

Effect of class size on age difference in scores

	Mean	90% credible interval	
γ_{00}	544.18	521.97	566.37
γ_{01}	-1.47	-2.02	-0.93
γ_{02}	0.23	-0.01	0.47
γ_{10}	-4.76	-5.87	-3.67
γ_{11}	0.10	-0.18	0.39

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\begin{aligned} \beta_{0k} = & \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i + \\ & \boxed{\gamma_{03}} \text{TeacherFemale}_k + \\ & \gamma_{04} \text{TeacherBlack}_k + \\ & \gamma_{05} \text{PropBlack}_k + \eta_{0k} \end{aligned}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \boxed{\gamma_{20}} + \boxed{\gamma_{21}} \text{TeacherFemale}_k + \eta_{2k}$$

$$\begin{aligned} \beta_{3k} = & \gamma_{30} + \gamma_{31} \text{TeacherBlack}_k + \\ & \gamma_{32} \text{PropBlack}_k + \eta_{3k} \end{aligned}$$

Baseline gender difference
in teacher effectiveness

Expected score difference between female
versus male students with male teacher

Expected score difference for female
students with female versus male teachers

	<i>Mean</i>	<i>90% credible interval</i>	
γ_{03}	-12.73	-34.86	9.44
γ_{20}	10.61	-7.43	28.56
γ_{21}	-0.61	-18.72	17.43

Two-level model: estimates

$$S_{ik} \sim \text{Norm}(\mu_{ik}, \sigma)$$

$$\mu_{ik} = \beta_{0k} + \beta_{1k} \text{Age}_i + \beta_{2k} \text{Female}_i + \beta_{3k} \text{Black}_i$$

$$\beta_{0k} = \gamma_{00} + \gamma_{01} \text{Size}_k + \gamma_{02} \text{Exp}_i +$$

$$\gamma_{03} \text{TeacherFemale}_k +$$

$$\boxed{\gamma_{04}} \text{TeacherBlack}_k +$$

$$\boxed{\gamma_{05}} \text{PropBlack}_k + \eta_{0k}$$

$$\beta_{1k} = \gamma_{10} + \gamma_{11} \text{Size}_k + \eta_{1k}$$

$$\beta_{2k} = \gamma_{20} + \gamma_{21} \text{TeacherFemale}_k + \eta_{2k}$$

$$\beta_{3k} = \boxed{\gamma_{30}} + \boxed{\gamma_{31}} \text{TeacherBlack}_k + \text{Race difference in teacher effectiveness (white students)}$$

$$\boxed{\gamma_{32}} \text{PropBlack}_k + \eta_{3k}$$

Effect of class racial composition (white students)

Baseline race difference in student scores

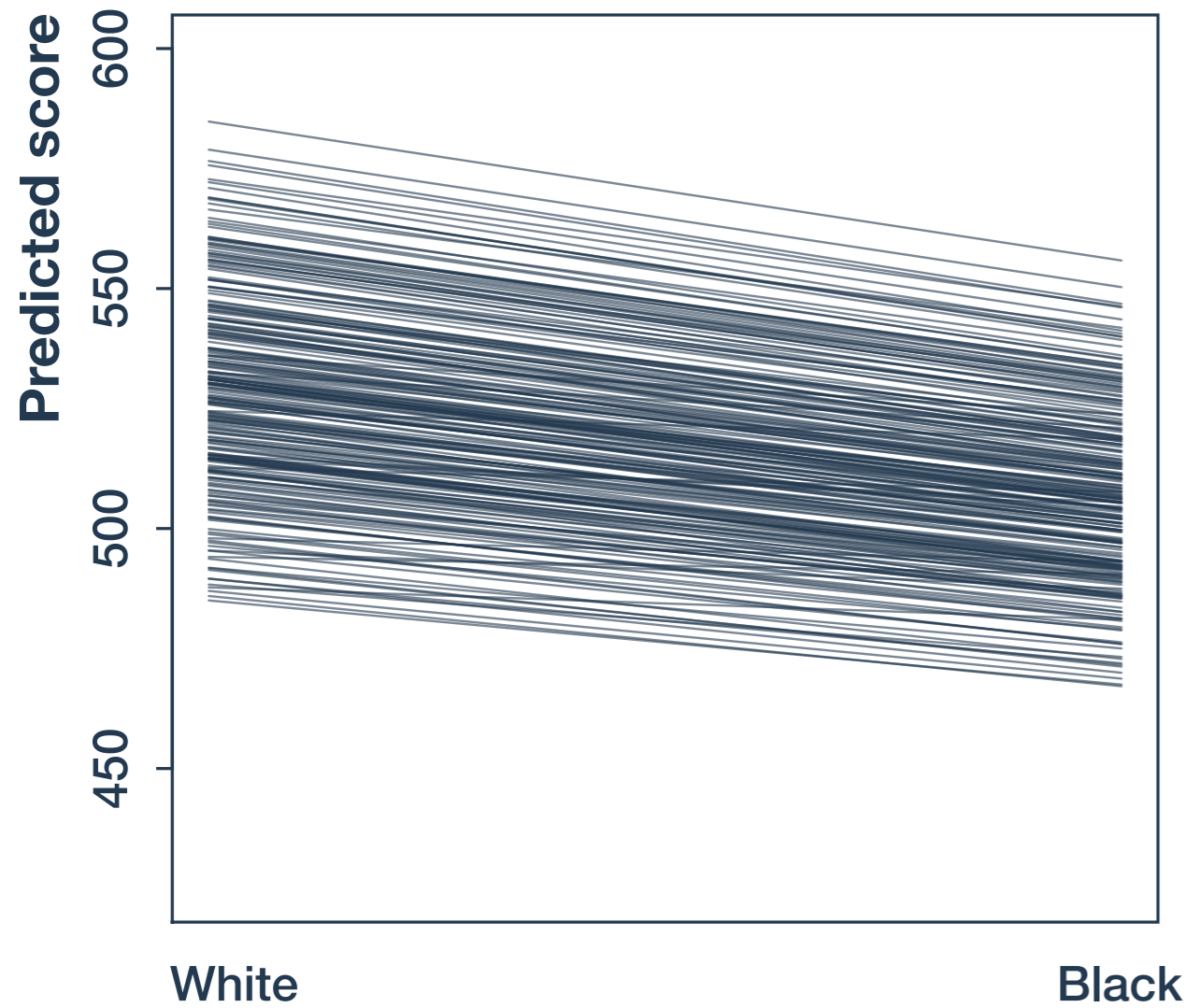
Added effect of having a Black teacher for Black (versus white) students

Added effect of class racial composition for Black (versus white) students

	Mean	90% credible interval	
γ_{04}	-7.96	-17.80	1.74
γ_{05}	-25.99	-38.44	-13.41
γ_{30}	-23.73	-29.22	-18.09
γ_{31}	13.97	4.37	23.78
γ_{32}	3.97	-10.53	18.72

Two-level model: estimates

White teachers



Black teachers

